

FRACTIONAL VECTOR-VALUED LITTLEWOOD–PALEY–STEIN THEORY FOR SEMIGROUPS

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ABSTRACT. We consider the fractional derivative of a general Poisson semigroup. With this fractional derivative, we define the generalized fractional Littlewood–Paley g -function for semigroups acting on L^p -spaces of functions with values in Banach spaces. We give a characterization of the classes of Banach spaces for which the fractional Littlewood–Paley g -function is bounded on L^p -spaces. We show that the class of Banach spaces is independent of the order of derivation and coincides with the classical (Lusin type/cotype) case. It is also shown that the same kind of results exist for the case of the fractional area function and the fractional g_λ^* -function on \mathbb{R}^n .

At last, we consider the relationship of the almost sure finiteness of the fractional Littlewood–Paley g -function, area function, and g_λ^* -function with the Lusin cotype property of the underlying Banach space. As a byproduct of the techniques developed, one can get some results of independent interest for vector-valued Calderón–Zygmund operators. For example, one can get the following characterization, a Banach space \mathbb{B} is UMD if and only if for some (or, equivalently, for every)

$$p \in [1, \infty), \quad \lim_{\varepsilon \rightarrow 0^+} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} dy \text{ exists a.e. } x \in \mathbb{R} \text{ for every } f \in L_{\mathbb{B}}^p(\mathbb{R}).$$

1. INTRODUCTION

In the last decade, a lot of attention has been devoted to the study of fractional laplacians, see [3, 14] and the references therein. On the other hand, several concepts of fractional derivatives have been developed in the literature since 19th century. Depending on the motivation of the researchers, these two objects can be different and even unrelated. However, when dealing with semigroups, it is clear that any definition of fractional derivative should have relation with the definition of fractional laplacian. Roughly speaking, a fractional derivative (with respect to t) of order “ α ” of the Poisson semigroup, $e^{-t\sqrt{\mathcal{L}}}$, of a certain differential operator \mathcal{L} , should be closely related to $\mathcal{L}^{\alpha/2}e^{-t\sqrt{\mathcal{L}}}$.

Segovia and Wheeden, see [11], motivated by some characterization of potential spaces on \mathbb{R}^n , introduced the following definition of “fractional derivative” ∂^α . Given $\alpha > 0$, let m be the smallest integer which strictly exceeds α . Let f be a reasonable nice function in $L_{\mathbb{B}}^p(\mathbb{R}^n)$. Then

$$\partial_t^\alpha \mathcal{P}_t f(x) = \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \mathcal{P}_{t+s}(f)(x) s^{m-\alpha-1} ds, \quad t > 0, x \in \mathbb{R}^n,$$

where Γ denotes the Gamma function and \mathcal{P}_t denotes the classical Poisson semigroup on \mathbb{R}^n . Observe that for reasonable good functions, $\partial_t^\alpha \mathcal{P}_t f(x) = e^{i\pi\alpha}(-\Delta)^{\alpha/2} \mathcal{P}_t f(x)$. In [11], the authors developed a satisfactory theory of euclidean square functions of Littlewood–Paley type in which the usual derivatives are substituted by these fractional derivatives.

It turns out that the notion of partial derivative considered by Segovia and Wheeden can be used in the case of general subordinated Poisson semigroups defined on a measure space $(\Omega, d\mu)$, see Section 2. Of course, without having a pointwise expression but just an identity in $L^p(\Omega)$. This fractional derivative has a nice behavior for iteration and for spectral decomposition. Then it is natural to ask whether results already known for classical derivatives are still true for the fractional derivative case. In this paper, we shall be concerned with several characterizations of Lusin type and Lusin cotype

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of Banach spaces by the boundedness of square functions defined by using the fractional derivatives. Now we explain briefly the concept of Lusin type and Lusin cotype.

The martingale type and cotype properties of a Banach space \mathbb{B} were introduced in the 1970's by G. Pisier, see [9, 10], in connection with the convexity and smoothness of the Banach space \mathbb{B} . If $M = (M_n)_{n \in \mathbb{N}}$ is a martingale defined on some probability space and with values in \mathbb{B} , the q -square function $S_q M$ is defined by $S_q M = \left(\sum_{n=1}^{\infty} \|M_n - M_{n-1}\|_{\mathbb{B}}^q \right)^{\frac{1}{q}}$. The Banach space \mathbb{B} is said to be of martingale cotype q , $2 \leq q < \infty$, if for every bounded $L_{\mathbb{B}}^p$ -martingale $M = (M_n)_{n \in \mathbb{N}}$ we have $\|S_q M\|_{L^p} \leq C_p \sup_n \|M_n\|_{L_{\mathbb{B}}^p}$, for some $1 < p < \infty$. The Banach space \mathbb{B} is said to be of martingale type q , $1 < q \leq 2$, when the reverse inequality holds for some $1 < p < \infty$. The martingale type and cotype properties do not depend on $1 < p < \infty$ for which the corresponding inequalities are satisfied. \mathbb{B} is of martingale cotype q if and only if its dual, \mathbb{B}^* , is of martingale type $q' = q/(q-1)$.

It is a common fact that results in probability theory have parallels in harmonic analysis. In this line of thought, Xu, see [15], defined the Lusin cotype and Lusin type properties for a Banach space \mathbb{B} as follows. Let f be a function in $L^1(\mathbb{T}, \mathbb{B})$, where \mathbb{T} denotes the one dimensional torus and $L^1(\mathbb{T}, \mathbb{B})$ stands for the Bochner–Lebesgue space of strong measurable \mathbb{B} -valued functions such that the scalar function $\|f\|_{\mathbb{B}}$ is integrable. Consider the generalized Littlewood–Paley g -function

$$g_q(f)(z) = \left(\int_0^1 (1-r)^q \|\partial_r P_r * f(z)\|_{\mathbb{B}}^q \frac{dr}{1-r} \right)^{\frac{1}{q}},$$

where $P_r(\theta)$ denotes the Poisson kernel. It is said that \mathbb{B} is of Lusin cotype q , $q \geq 2$, if for some $1 < p < \infty$ we have $\|g_q(f)\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L_{\mathbb{B}}^p(\mathbb{T})}$, and \mathbb{B} is of Lusin type q , $1 \leq q \leq 2$, if for some $1 < p < \infty$ we have $\|f\|_{L_{\mathbb{B}}^p(\mathbb{T})} \leq C_p \left(\|\hat{f}(0)\|_{\mathbb{B}} + \|g_q(f)\|_{L^p(\mathbb{T})} \right)$.

The Lusin cotype and Lusin type properties do not depend on $p \in (1, \infty)$, see [15, 8]. Moreover, a Banach space \mathbb{B} is of Lusin cotype q (Lusin type q) if and only if \mathbb{B} is of martingale cotype q (martingale type q), see [15, Theorem 3.1].

Martínez, Torrea and Xu, see [7], extended the results in [15] to subordinated Poisson semigroup $\{\mathcal{P}_t\}_{t \geq 0}$ of a general symmetric diffusion Markovian semigroup $\{\mathcal{T}_t\}_{t \geq 0}$. That is, a family of linear operators defined on $L^p(\Omega, d\mu)$ over a measure space $(\Omega, d\mu)$ satisfying the semigroup properties

- $\mathcal{T}_0 = \text{Id}$, $\mathcal{T}_t \mathcal{T}_s = \mathcal{T}_{t+s}$.
- $\|\mathcal{T}_t\|_{L^p \rightarrow L^p} \leq 1 \quad \forall p \in [1, \infty]$,
- $\lim_{t \rightarrow 0} \mathcal{T}_t f = f \quad \text{in } L^2 \quad \forall f \in L^2$,
- $\mathcal{T}_t^* = \mathcal{T}_t \quad \text{on } L^2 \quad \text{and}$
- $\mathcal{T}_t f \geq 0 \quad \text{if } f \geq 0, \quad \mathcal{T}_t 1 = 1$.

The subordinated Poisson semigroup $\{\mathcal{P}_t\}_{t \geq 0}$ (again a symmetric diffusion semigroup, see [12]) is defined as

$$(1.1) \quad \mathcal{P}_t f = \frac{t}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} \mathcal{T}_u f du.$$

Being positive operators, \mathcal{T}_t and \mathcal{P}_t have straightforward norm-preserving extensions to $L_{\mathbb{B}}^p(\Omega)$ for every Banach space \mathbb{B} , where $L_{\mathbb{B}}^p(\Omega)$ denotes the usual Bochner–Lebesgue L^p -space of \mathbb{B} -valued functions defined on Ω . Let g_1^q be the generalized Littlewood–Paley g -function defined by

$$g_1^q(f)(x) = \left(\int_0^{\infty} \|t \partial_t \mathcal{P}_t f(x)\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad \forall f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\Omega).$$

The results in [7] are as follows.

Theorem 1.1. *Given a Banach space \mathbb{B} and $2 \leq q < \infty$, the following statements are equivalent:*

- (i) \mathbb{B} is of Lusin cotype q .

- (ii) For every subordinated Poisson semigroup $\{\mathcal{P}_t\}_{t \geq 0}$ and for every (or, equivalently, for some) $p \in (1, \infty)$, there is a constant C such that $\|g_1^q(f)\|_{L^p(\Omega)} \leq C\|f\|_{L_{\mathbb{B}}^p(\Omega)}$, for every $f \in L_{\mathbb{B}}^p(\Omega)$.

Let $\mathbb{E}_0 \subset L^2(\Omega)$ be the subspace of all h such that $\mathcal{P}_t(h) = h$ for all $t \geq 0$. Let $E_0 : L^2(\Omega) \rightarrow \mathbb{E}_0$ be the orthogonal projection. E_0 extends to a contractive projection on $L^p(\Omega)$ for every $1 \leq p < \infty$. $E_0(L^p(\Omega))$ is exactly the fix point space of $\{\mathcal{P}_t\}_{t \geq 0}$ on $L^p(\Omega)$, see [12]. Moreover, for any Banach space \mathbb{B} , E_0 extends to a contractive projection on $L_{\mathbb{B}}^p(\Omega)$ for every $1 \leq p < \infty$ and $E_0(L_{\mathbb{B}}^p(\Omega))$ is again the fix point space of $\{\mathcal{P}_t\}_{t \geq 0}$ considered as a semigroup on $L_{\mathbb{B}}^p(\Omega)$. In the particular case on \mathbb{R}^n , $\mathbb{E}_0 = 0$ and so $E_0(L_{\mathbb{B}}^p(\mathbb{R}^n)) = \{0\}$. In the sequel, we shall use the same symbol E_0 to denote any of these contractive projections.

Theorem 1.2. *Given a Banach space \mathbb{B} and $1 < q \leq 2$, the following statements are equivalent:*

- (i) \mathbb{B} is of Lusin type q .
- (ii) For every subordinated Poisson semigroup $\{\mathcal{P}_t\}_{t \geq 0}$ and for every (or, equivalently, for some) $p \in (1, \infty)$, there is a constant C such that $\|f\|_{L_{\mathbb{B}}^p(\Omega)} \leq C\left(\|E_0(f)\|_{L_{\mathbb{B}}^p(\Omega)} + \|g_1^q(f)\|_{L^p(\Omega)}\right)$, for every $f \in L_{\mathbb{B}}^p(\Omega)$.

Theorem 1.3. *Given a Banach space \mathbb{B} and $2 \leq q < \infty$, the following statements are equivalent:*

- (i) \mathbb{B} is of Lusin cotype q .
- (ii) For any $f \in L_{\mathbb{B}}^1(\mathbb{T})$, $g_q(f)(z) < \infty$ for almost every $z \in \mathbb{T}$.
- (iii) For any $f \in L_{\mathbb{B}}^1(\mathbb{R}^n)$, $g_1^q(f)(x) < \infty$ for almost every $x \in \mathbb{R}^n$.

As we said before, our goal is to characterize Lusin cotype and Lusin type properties of Banach spaces when the standard derivative is substitute by the fractional derivative. Parallel to Segovia and Wheeden, we define

$$(1.2) \quad \partial_t^\alpha \mathcal{P}_t f = \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \mathcal{P}_{t+s}(f) s^{m-\alpha-1} ds, \quad t > 0,$$

where m is the smallest integer which strictly exceeds α . In Section 2, we shall see that for any $f \in L^p(\Omega)$, this partial derivative is well defined and then we are allowed to consider the following “fractional Littlewood–Paley g -function”

$$(1.3) \quad g_\alpha^q(f) = \left(\int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\Omega), \alpha > 0.$$

The results in this paper can be classified in three types:

- Theorems which generalize the results in [7] for the case of fractional derivatives (Theorem A and Theorem B).
- New theorems involving area functions and g_λ^* functions on \mathbb{R}^n (Theorem 5.3 – Theorem 5.6).
- New results for characterizations of Lusin cotype through almost everywhere finiteness (Theorem C).

In our opinion, it is worth to mention that the proof of Theorem C contains some new ideas that can be applied to a huge class of operators. Roughly, the method used in the proof is the following. If an operator T with a Calderón–Zygmund kernel is a.e. pointwise finite ($Tf(x) < \infty$) for any function f in $L^{p_0}(\mathbb{R}^n)$ and some $p_0 \in [1, \infty)$, then T is bounded from $L^1(\mathbb{R}^n)$ into weak- $L^1(\mathbb{R}^n)$. This philosophy can be translated to the vector-valued case and we can get results like the one presented in Theorem D.

Now we list our main theorems.

Theorem A. *Given a Banach space \mathbb{B} and $2 \leq q < \infty$, the following statements are equivalent:*

- (i) \mathbb{B} is of Lusin cotype q .
- (ii) For every symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t \geq 0}$ with subordinated semigroup $\{\mathcal{P}_t\}_{t \geq 0}$, for every (or, equivalently, for some) $p \in (1, \infty)$, and for every (or, equivalently, for some) $\alpha > 0$,

there is a constant C such that

$$\|g_\alpha^q(f)\|_{L^p(\Omega)} \leq C\|f\|_{L_\mathbb{B}^p(\Omega)}, \quad \forall f \in L_\mathbb{B}^p(\Omega).$$

Theorem B. Given a Banach space \mathbb{B} and $1 < q \leq 2$, the following statements are equivalent:

- (i) \mathbb{B} is of Lusin type q .
- (ii) For every symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t \geq 0}$ with subordinated semigroup $\{\mathcal{P}_t\}_{t \geq 0}$, for every (or, equivalently, for some) $p \in (1, \infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant C such that

$$\|f\|_{L_\mathbb{B}^p(\Omega)} \leq C \left(\|E_0(f)\|_{L_\mathbb{B}^p(\Omega)} + \|g_\alpha^q(f)\|_{L^p(\Omega)} \right), \quad \forall f \in L_\mathbb{B}^p(\Omega).$$

On the particular Lebesgue measure space (\mathbb{R}^n, dx) , we have the following theorems.

Theorem C. Given a Banach space \mathbb{B} , $2 \leq q < \infty$, the following statements are equivalent:

- (i) \mathbb{B} is of Lusin cotype q .
- (ii) For every (or, equivalently, for some) $p \in [1, \infty)$ and for every (or, equivalently, for some) $\alpha > 0$, $g_\alpha^q(f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, for every $f \in L_\mathbb{B}^p(\mathbb{R}^n)$.
- (iii) For every (or, equivalently, for some) $p \in [1, \infty)$ and for every (or, equivalently, for some) $\alpha > 0$, $S_\alpha^q(f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, for every $f \in L_\mathbb{B}^p(\mathbb{R}^n)$.
- (iv) For every (or, equivalently, for some) $p \in [q, \infty)$ and for every (or, equivalently, for some) $\alpha > 0$, $g_{\lambda, \alpha}^{q,*}(f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, for every $f \in L_\mathbb{B}^p(\mathbb{R}^n)$.

Theorem D. Given a Banach space \mathbb{B} , the following statements are equivalent:

- (i) \mathbb{B} is UMD.
- (ii) For every (or, equivalently, for some) $p \in [1, \infty)$,

$$\lim_{\varepsilon \rightarrow 0^+} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} dy \text{ exists a.e. } x \in \mathbb{R}, \text{ for every } f \in L_\mathbb{B}^p(\mathbb{R}).$$

We want to mention that in [4] T. Hytönen extended some results in [7] to the case of appropriated stochastic integrals. On the other hand we think that this paper contains some new conical square function estimates in the sense of [5].

The paper is organized as follows. In Section 2, we present a systematic study of some properties related to the fractional derivatives for general Poisson semigroups. Section 3 is devoted to the analysis of several relations between the fractional Littlewood–Paley g -functions, some of them for general Poisson semigroups, while others are for Poisson semigroups on \mathbb{R}^n . In this case, we need the theory of Calderón–Zygmund as a fundamental tool. Section 4 contains the proofs of Theorem A and Theorem B. Section 5 is devoted to discuss the similar results for the fractional area function and the fractional g_λ^* -function on \mathbb{R}^n . Section 6 is devoted to the proof of Theorem C. Finally we prove Theorem D in Section 7.

Throughout this paper, the letter C will denote a positive constant which may change from one instance to another and depend on the parameters involved. We will make a frequent use, without mentioning it in relevant places, of the fact that for a positive A and a non-negative a ,

$$\sup_{t > 0} t^a \exp(-At) = C_{a,A} < \infty.$$

2. FRACTIONAL DERIVATIVES

In this section, we shall consider the general symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t \geq 0}$ defined on $L^p(\Omega)$. Given such a semigroup $\{\mathcal{T}_t\}_{t \geq 0}$, we consider its subordinated semigroup $\{\mathcal{P}_t\}_{t \geq 0}$ defined as in (1.1).

Theorem 2.1. *Given a Banach space \mathbb{B} , $1 \leq p \leq \infty$, $\alpha > 0$, and $t > 0$, $\partial_t^\alpha \mathcal{P}_t f$ is well defined as a function in $L_{\mathbb{B}}^p(\Omega)$ for any $f \in L_{\mathbb{B}}^p(\Omega)$. Moreover, there exists a constant C_α such that*

$$(2.1) \quad \|\partial_t^\alpha \mathcal{P}_t f\|_{L_{\mathbb{B}}^p(\Omega)} \leq \frac{C_\alpha}{t^\alpha} \|f\|_{L_{\mathbb{B}}^p(\Omega)}, \quad \forall f \in L_{\mathbb{B}}^p(\Omega).$$

Proof. Firstly, let us consider the case $\alpha = m$, $m = 1, 2, \dots$. We know that, for any $m = 1, 2, \dots$, there exist constants C_m such that

$$\partial_t^m \left(\frac{t}{\sqrt{u}} e^{-\frac{t^2}{4u}} \right) \leq C_m \frac{1}{(\sqrt{u})^m} e^{-\frac{t^2}{4u}}.$$

Then, by using formula (1.1), we have

$$(2.2) \quad \begin{aligned} \|\partial_t^m \mathcal{P}_t f\|_{L_{\mathbb{B}}^p(\Omega)} &\leq C \int_0^\infty \left| \partial_t^m \left(\frac{t}{\sqrt{u}} e^{-\frac{t^2}{4u}} \right) \right| \|\mathcal{T}_u f\|_{L_{\mathbb{B}}^p(\Omega)} \frac{du}{u} \\ &\leq C_m \int_0^\infty \frac{1}{(\sqrt{u})^m} e^{-\frac{t^2}{4u}} \frac{du}{u} \|f\|_{L_{\mathbb{B}}^p(\Omega)} = \frac{C_m}{t^m} \|f\|_{L_{\mathbb{B}}^p(\Omega)}. \end{aligned}$$

So we have proved (2.1) when α is integer. Therefore, given $\alpha > 0$, we have

$$(2.3) \quad \begin{aligned} \|\partial_t^\alpha \mathcal{P}_t f\|_{L_{\mathbb{B}}^p(\Omega)} &= \left\| \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \mathcal{P}_{t+s}(f) s^{m-\alpha-1} ds \right\|_{L_{\mathbb{B}}^p(\Omega)} \\ &\leq \frac{C_m}{\Gamma(m-\alpha)} \|f\|_{L_{\mathbb{B}}^p(\Omega)} \int_0^\infty \frac{1}{(t+s)^m} s^{m-\alpha-1} ds \\ &= \frac{C_m}{\Gamma(m-\alpha)} \mathbf{B}(m-\alpha, \alpha) \frac{\|f\|_{L_{\mathbb{B}}^p(\Omega)}}{t^\alpha} = C_\alpha \frac{\|f\|_{L_{\mathbb{B}}^p(\Omega)}}{t^\alpha}, \end{aligned}$$

where \mathbf{B} denotes the Beta function, see [6]. □

Observe that by estimate (2.2), we can perform integration by parts in the formula (1.2). In particular, the formula (1.2) is valid for α being integer.

Theorem 2.2. *Given a Banach space \mathbb{B} and $0 < \beta < \gamma$, we have*

$$(2.4) \quad \partial_t^\beta \mathcal{P}_t f = \frac{e^{-i\pi(\gamma-\beta)}}{\Gamma(\gamma-\beta)} \int_0^\infty \partial_t^\gamma \mathcal{P}_{t+s}(f) s^{\gamma-\beta-1} ds, \quad \forall f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\Omega).$$

Proof. Assume that $f \in L_{\mathbb{B}}^p(\Omega)$ for some $1 \leq p \leq \infty$, by changing variables and Fubini's theorem, we have the following computation as in (2.3)

$$(2.5) \quad \begin{aligned} \int_0^\infty \partial_t^\gamma \mathcal{P}_{t+s}(f) s^{\gamma-\beta-1} ds &= \int_0^\infty \frac{e^{-i\pi(k-\gamma)}}{\Gamma(k-\gamma)} \int_0^\infty \partial_t^k \mathcal{P}_{t+s+u}(f) u^{k-\gamma-1} du s^{\gamma-\beta-1} ds \\ &= \frac{e^{-i\pi(k-\gamma)}}{\Gamma(k-\gamma)} \int_0^\infty \int_s^\infty \partial_t^k \mathcal{P}_{t+\bar{u}}(f) (\bar{u}-s)^{k-\gamma-1} s^{\gamma-\beta-1} d\bar{u} ds \\ &= \frac{e^{-i\pi(k-\gamma)} \mathbf{B}(k-\gamma, \gamma-\beta)}{\Gamma(k-\gamma)} \int_0^\infty \partial_t^k \mathcal{P}_{t+\bar{u}}(f) \bar{u}^{k-\beta-1} d\bar{u}, \end{aligned}$$

where k is the smallest integer which is bigger than γ . By (2.2), we know that we can integrate by parts in the last integral of (2.5). Let m be the smallest integer which is bigger than β . Then by integrating by parts $k-m$ times, we obtain

$$\begin{aligned} &\int_0^\infty \partial_t^\gamma \mathcal{P}_{t+s}(f) s^{\gamma-\beta-1} ds \\ &= \frac{\mathbf{B}(k-\gamma, \gamma-\beta) e^{-i\pi(m-\gamma)}}{\Gamma(k-\gamma)} (k-\beta-1) \cdots (m-\beta) \int_0^\infty \partial_t^m \mathcal{P}_{t+\bar{u}}(f) \bar{u}^{m-\beta-1} d\bar{u} \\ &= e^{-i\pi(\gamma-\beta)} \Gamma(\gamma-\beta) \partial_t^\beta \mathcal{P}_t f. \end{aligned}$$

Hence we get (2.4). □

Theorem 2.3. *Given a Banach space \mathbb{B} and $\alpha, \beta > 0$, $\partial_t^\alpha \left(\partial_t^\beta \mathcal{P}_t f \right)$ can be defined as*

$$(2.6) \quad \partial_t^\alpha \left(\partial_t^\beta \mathcal{P}_t f \right) = \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \left(\partial_{t+s}^\beta \mathcal{P}_{t+s} f \right) s^{m-\alpha-1} ds, \quad \forall f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\Omega),$$

where m is the smallest integer which is bigger than α . Then

$$(2.7) \quad \partial_t^\alpha \left(\partial_t^\beta \mathcal{P}_t f \right) = \partial_t^{\alpha+\beta} \mathcal{P}_t f, \quad \forall f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\Omega).$$

Proof. For any $f \in L_{\mathbb{B}}^p(\Omega)$ for some $1 \leq p \leq \infty$, by (1.2) and Theorem 2.1 we have the following computation for the latter of (2.6):

$$(2.8) \quad \begin{aligned} & \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \left(\partial_{t+s}^\beta \mathcal{P}_{t+s} f \right) s^{m-\alpha-1} ds \\ &= \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)} \int_0^\infty \partial_t^m \left(\int_0^\infty \partial_{t+s}^k \mathcal{P}_{t+s+u}(f) u^{k-\beta-1} du \right) s^{m-\alpha-1} ds, \end{aligned}$$

where k is the smallest integer which is bigger than β . For any fixed $s \in (0, \infty)$, $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \subset (0, \infty)$ for some $t_0 \in (0, \infty)$, and $\varepsilon > 0$, by (2.1) we have

$$(2.9) \quad \begin{aligned} & \left\| \partial_t^m \left(\partial_{t+s}^k \mathcal{P}_{t+s+u}(f) u^{k-\beta-1} \right) \right\|_{L_{\mathbb{B}}^p(\Omega)} = \left\| \partial_t^{m+k} \mathcal{P}_{t+s+u}(f) \right\|_{L_{\mathbb{B}}^p(\Omega)} u^{k-\beta-1} \\ & \leq \frac{C}{(t+s+u)^{m+k}} u^{k-\beta-1} \|f\|_{L_{\mathbb{B}}^p(\Omega)} \leq \frac{C}{(t_0 - \varepsilon + s + u)^{m+k}} u^{k-\beta-1} \|f\|_{L_{\mathbb{B}}^p(\Omega)}, \end{aligned}$$

for any $1 \leq p \leq \infty$. And

$$(2.10) \quad \begin{aligned} & \int_0^\infty \left| \frac{u^{k-\beta-1}}{(t_0 - \varepsilon + s + u)^{m+k}} \right| du \|f\|_{L_{\mathbb{B}}^p(\Omega)} \\ &= \left(\int_0^{t_0 - \varepsilon + s} \left| \frac{u^{k-\beta-1}}{(t_0 - \varepsilon + s + u)^{m+k}} \right| du + \int_{t_0 - \varepsilon + s}^\infty \left| \frac{u^{k-\beta-1}}{(t_0 - \varepsilon + s + u)^{m+k}} \right| du \right) \|f\|_{L_{\mathbb{B}}^p(\Omega)} \\ &\leq C \frac{1}{(t_0 - \varepsilon + s)^{\beta+m}} \|f\|_{L_{\mathbb{B}}^p(\Omega)} < \infty. \end{aligned}$$

Combining (2.9) and (2.10), we know that $\partial_t^m \left(\partial_{t+s}^k \mathcal{P}_{t+s+u}(f) u^{k-\beta-1} \right)$ is controlled by an integrable function. Hence we can interchange the order of the inner integration and the partial derivative ∂_t^m in (2.8) to obtain

$$(2.11) \quad \begin{aligned} & \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \left(\partial_{t+s}^\beta \mathcal{P}_{t+s} f \right) s^{m-\alpha-1} ds \\ &= \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)} \int_0^\infty \int_0^\infty \partial_t^m \partial_{t+s}^k \mathcal{P}_{t+s+u}(f) u^{k-\beta-1} du s^{m-\alpha-1} ds \\ &= \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)} \int_0^\infty \int_0^\infty \partial_t^{m+k} \mathcal{P}_{t+s+u}(f) u^{k-\beta-1} du s^{m-\alpha-1} ds \\ &= \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)} \int_0^\infty \int_s^\infty \partial_t^{m+k} \mathcal{P}_{t+w}(f) (w-s)^{k-\beta-1} dw s^{m-\alpha-1} ds \\ &= \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)} \int_0^\infty \int_0^w \partial_t^{m+k} \mathcal{P}_{t+w}(f) (w-s)^{k-\beta-1} s^{m-\alpha-1} ds dw \\ &= \frac{e^{-i\pi(m+k-\alpha-\beta)} \mathbf{B}(m-\alpha, k-\beta)}{\Gamma(m-\alpha)\Gamma(k-\beta)} \int_0^\infty \partial_t^{m+k} \mathcal{P}_{t+w}(f) w^{k+m-\alpha-\beta-1} dw \\ &= \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m+k-\alpha-\beta)} \int_0^\infty \partial_t^{m+k} \mathcal{P}_{t+w}(f) w^{k+m-\alpha-\beta-1} dw. \end{aligned}$$

Since $m-1 \leq \alpha < m$ and $k-1 \leq \beta < k$, $m+k-2 \leq \alpha+\beta < m+k$. If $m+k-1 \leq \alpha+\beta < m+k$, we have

$$(2.12) \quad \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m+k-\alpha-\beta)} \int_0^\infty \partial_t^{m+k} \mathcal{P}_{t+w}(f) w^{k+m-\alpha-\beta-1} dw = \partial_t^{\alpha+\beta} \mathcal{P}_t f.$$

If $m+k-2 \leq \alpha+\beta < m+k-1$, then integrating by parts, we get

$$(2.13) \quad \begin{aligned} \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m+k-\alpha-\beta)} \int_0^\infty \partial_t^{m+k} \mathcal{P}_{t+w}(f) w^{k+m-\alpha-\beta-1} dw \\ = \frac{e^{-i\pi(m+k-1-\alpha-\beta)}}{\Gamma(m+k-1-\alpha-\beta)} \int_0^\infty \partial_t^{m+k-1} \mathcal{P}_{t+w}(f) w^{k+m-\alpha-\beta-2} dw = \partial_t^{\alpha+\beta} \mathcal{P}_t f. \end{aligned}$$

So, combining (2.8) and (2.11)–(2.12), we get

$$\frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \left(\partial_{t+s}^\beta \mathcal{P}_{t+s} f \right) s^{m-\beta-1} ds = \partial_t^{\alpha+\beta} \mathcal{P}_t f,$$

for any $f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\Omega)$. □

Write the spectral decomposition of the semigroup $\{\mathcal{P}_t\}_{t \geq 0}$: for any $f \in L^2(\Omega)$

$$\mathcal{P}_t f = \int_0^\infty e^{-\lambda t} dE_f(\lambda),$$

where $E(\lambda)$ is a resolution of the identity. Thus

$$(2.14) \quad \partial_t^k \mathcal{P}_t f = e^{-i\pi k} \int_{0+}^\infty \lambda^k e^{-\lambda t} dE_f(\lambda), \quad k = 1, 2, \dots$$

We have the following proposition.

Proposition 2.4. *Let $f \in L^2(\Omega)$ and $0 < \alpha < \infty$. We have*

$$(2.15) \quad \partial_t^\alpha \mathcal{P}_t f = e^{-i\pi\alpha} \int_{0+}^\infty \lambda^\alpha e^{-\lambda t} dE_f(\lambda).$$

Proof. By (1.2) and (2.14), we have

$$(2.16) \quad \partial_t^\alpha \mathcal{P}_t f = \frac{e^{-i\pi\alpha}}{\Gamma(k-\alpha)} \int_0^\infty \int_{0+}^\infty \lambda^k e^{-(t+s)\lambda} dE_f(\lambda) s^{k-\alpha-1} ds,$$

where k is the smallest integer which is bigger than α . Then $\int_0^\infty \int_{0+}^\infty \lambda^k e^{-(t+s)\lambda} |dE(\lambda)| s^{k-\alpha-1} ds$ is absolutely convergent. And by Theorem 2.1, we know that the integral in (1.2) is absolutely convergent in $L^2(\Omega)$. So by (2.16), we get

$$\begin{aligned} \langle \partial_t^\alpha \mathcal{P}_t f, g \rangle &= \left\langle \frac{e^{-i\pi\alpha}}{\Gamma(k-\alpha)} \int_0^\infty \int_{0+}^\infty \lambda^k e^{-(t+s)\lambda} dE_f(\lambda) s^{k-\alpha-1} ds, g \right\rangle \\ &= \frac{e^{-i\pi\alpha}}{\Gamma(k-\alpha)} \int_0^\infty \left\langle \int_{0+}^\infty \lambda^k e^{-(t+s)\lambda} dE_f(\lambda), g \right\rangle s^{k-\alpha-1} ds \\ &= \frac{e^{-i\pi\alpha}}{\Gamma(k-\alpha)} \int_0^\infty \int_{0+}^\infty \lambda^k e^{-(t+s)\lambda} dE_{\langle f, g \rangle}(\lambda) s^{k-\alpha-1} ds \\ &= \frac{e^{-i\pi\alpha}}{\Gamma(k-\alpha)} \int_{0+}^\infty \int_0^\infty \lambda^k e^{-(t+s)\lambda} s^{k-\alpha-1} ds dE_{\langle f, g \rangle}(\lambda) \\ &= \left\langle e^{-i\pi\alpha} \int_{0+}^\infty \lambda^\alpha e^{-t\lambda} dE_f(\lambda), g \right\rangle, \quad \forall g \in L^2(\Omega). \end{aligned}$$

Hence we get (2.15). □

3. TECHNICAL RESULTS FOR LITTLEWOOD–PALEY g -FUNCTION

In this section, we will give some properties of the fractional Littlewood–Paley g -function.

Proposition 3.1. *Given a Banach space \mathbb{B} , $1 < q < \infty$, and $0 < \beta < \gamma$, there exists a constant C such that*

$$(3.1) \quad g_\beta^q(f) \leq C g_\gamma^q(f), \quad \forall f \in \bigcup_{1 \leq p \leq \infty} L_\mathbb{B}^p(\Omega).$$

Proof. Assume that $f \in L_\mathbb{B}^p(\Omega)$ for some $1 \leq p \leq \infty$. By Theorem 2.2 and Hölder's inequality, we have

$$\begin{aligned} \|\partial_t^\beta \mathcal{P}_t f\|_\mathbb{B} &\leq \frac{1}{\Gamma(\gamma - \beta)} \int_t^\infty \|\partial_s^\gamma \mathcal{P}_s f\|_\mathbb{B} (s - t)^{\gamma - \beta - 1} ds \\ &\leq \frac{1}{\Gamma(\gamma - \beta)} \left(\int_t^\infty \|\partial_s^\gamma \mathcal{P}_s f\|_\mathbb{B}^q (s - t)^{\gamma - \beta - 1} s^{\gamma(q-1)} ds \right)^{\frac{1}{q}} \left(\int_t^\infty (s - t)^{\gamma - \beta - 1} s^{-\gamma} ds \right)^{\frac{1}{q'}. \end{aligned}$$

By changing variables, we have

$$\begin{aligned} \int_t^\infty (s - t)^{\gamma - \beta - 1} s^{-\gamma} ds &= \int_t^\infty \left(1 - \frac{t}{s}\right)^{\gamma - \beta - 1} \left(\frac{t}{s}\right)^{\beta + 1} t^{-\beta - 1} ds \\ &= t^{-\beta} \int_0^1 (1 - u)^{\gamma - \beta - 1} u^{\beta - 1} du = t^{-\beta} \mathbf{B}(\gamma - \beta, \beta). \end{aligned}$$

So we have

$$\begin{aligned} (3.2) \quad \|\partial_t^\beta \mathcal{P}_t f\|_\mathbb{B} &\leq \frac{1}{\Gamma(\gamma - \beta)} (t^{-\beta} \mathbf{B}(\gamma - \beta, \beta))^{\frac{1}{q'}} \left(\int_t^\infty \|\partial_s^\gamma \mathcal{P}_s f\|_\mathbb{B}^q (s - t)^{\gamma - \beta - 1} s^{\gamma(q-1)} ds \right)^{\frac{1}{q}} \\ &= \frac{(\mathbf{B}(\gamma - \beta, \beta))^{\frac{1}{q'}}}{\Gamma(\gamma - \beta)} (t^{-\beta})^{\frac{1}{q'}} \left(\int_t^\infty \|\partial_s^\gamma \mathcal{P}_s f\|_\mathbb{B}^q (s - t)^{\gamma - \beta - 1} s^{\gamma(q-1)} ds \right)^{\frac{1}{q}}. \end{aligned}$$

Using Fubini's theorem, by (3.2) we get

$$\begin{aligned} \int_0^\infty \|t^\beta \partial_t^\beta \mathcal{P}_t f\|_\mathbb{B}^q \frac{dt}{t} &\leq \frac{(\mathbf{B}(\gamma - \beta, \beta))^{\frac{q}{q'}}}{\Gamma(\gamma - \beta)^q} \int_0^\infty t^{\beta q} (t^{-\beta})^{\frac{q}{q'}} \int_t^\infty \|\partial_s^\gamma \mathcal{P}_s f\|_\mathbb{B}^q (s - t)^{\gamma - \beta - 1} s^{\gamma(q-1)} ds \frac{dt}{t} \\ &= \frac{(\mathbf{B}(\gamma - \beta, \beta))^{q-1}}{\Gamma(\gamma - \beta)^q} \int_0^\infty s^{\gamma(q-1)} \|\partial_s^\gamma \mathcal{P}_s f\|_\mathbb{B}^q \int_0^s t^{\beta-1} (s - t)^{\gamma - \beta - 1} dt ds \\ &= \left(\frac{\Gamma(\beta)}{\Gamma(\gamma)} \right)^q \int_0^\infty \|s^\gamma \partial_s^\gamma \mathcal{P}_s f\|_\mathbb{B}^q \frac{ds}{s}. \end{aligned}$$

Hence we get the inequality (3.1) with the constant $C = \frac{\Gamma(\beta)}{\Gamma(\gamma)}$. \square

In the following, we shall need the theory of Calderón–Zygmund on \mathbb{R}^n . So we should recall briefly the definition of the Calderón–Zygmund operator. Given two Banach spaces \mathbb{B}_1 and \mathbb{B}_2 , let T be a linear operator. Then we call that T is a Calderón–Zygmund operator on \mathbb{R}^n , with associated Calderón–Zygmund kernel K if T maps $L_{c, \mathbb{B}_1}^\infty$, the space of the essentially bounded \mathbb{B}_1 -valued functions on \mathbb{R}^n with compact support, into the space of \mathbb{B}_2 -valued and strongly measurable functions on \mathbb{R}^n , and for any function $f \in L_{c, \mathbb{B}_1}^\infty$ we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad \text{a.e. } x \in \mathbb{R}^n \text{ outside the support of } f,$$

where the kernel $K(x, y)$ is a regular kernel, that is, $K(x, y) \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ satisfies $\|K(x, y)\| \leq C \frac{1}{|x - y|^n}$ and $\|\nabla_x K(x, y)\| + \|\nabla_y K(x, y)\| \leq C \frac{1}{|x - y|^{n+1}}$, for any $x, y \in \mathbb{R}^n$ and $x \neq y$, where as usual $\nabla_x = (\partial_{x_1}, \dots, \partial_{x_n})$.

Let us recall the \mathbb{B} -valued BMO and H^1 spaces on \mathbb{R}^n . It is well known that

$$BMO_{\mathbb{B}}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathbb{B}, \text{loc}}(\mathbb{R}^n) : \sup_{\text{cubes } Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \left\| f(x) - \frac{1}{|Q|} \int_Q f(y) dy \right\|_{\mathbb{B}} dx < \infty \right\}.$$

The \mathbb{B} -valued H^1 space is defined in the atomic sense. We say that a function $a \in L^{\infty}_{\mathbb{B}}(\mathbb{R}^n)$ is a \mathbb{B} -valued atom if there exists a cube $Q \subset \mathbb{R}^n$ containing the support of a , and such that $\|a\|_{L^{\infty}_{\mathbb{B}}(\mathbb{R}^n)} \leq |Q|^{-1}$ and $\int_Q a(x) dx = 0$. Then, we can define $H^1_{\mathbb{B}}(\mathbb{R}^n)$ as

$$H^1_{\mathbb{B}}(\mathbb{R}^n) = \left\{ f : f = \sum_i \lambda_i a_i, \text{ } a_i \text{ are } \mathbb{B}\text{-valued atoms and } \sum_i |\lambda_i| < \infty \right\}.$$

We define $\|f\|_{H^1_{\mathbb{B}}(\mathbb{R}^n)} = \inf \left\{ \sum_i |\lambda_i| \right\}$, where the infimum runs over all those such decompositions.

Remark 3.2. [7, Theorem 4.1] Given a pair of Banach spaces \mathbb{B}_1 and \mathbb{B}_2 , let T be a Calderón-Zygmund operator on \mathbb{R}^n with regular vector-valued kernel. Then the following statements are equivalent:

- (i) T maps $L^{\infty}_{c, \mathbb{B}_1}(\mathbb{R}^n)$ into $BMO_{\mathbb{B}_2}(\mathbb{R}^n)$.
- (ii) T maps $H^1_{\mathbb{B}_1}(\mathbb{R}^n)$ into $L^1_{\mathbb{B}_2}(\mathbb{R}^n)$.
- (iii) T maps $L^p_{\mathbb{B}_1}(\mathbb{R}^n)$ into $L^p_{\mathbb{B}_2}(\mathbb{R}^n)$ for any (or, equivalently, for some) $p \in (1, \infty)$.
- (iv) T maps $BMO_{c, \mathbb{B}_1}(\mathbb{R}^n)$ into $BMO_{\mathbb{B}_2}(\mathbb{R}^n)$.
- (v) T maps $L^1_{\mathbb{B}_1}(\mathbb{R}^n)$ into $L^{1, \infty}_{\mathbb{B}_2}(\mathbb{R}^n)$.

Proposition 3.3. *Given a Banach space \mathbb{B} , $1 < q < \infty$, and $0 < \alpha < \infty$, $g^q_{\alpha}(f)$ can be expressed as an $L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$ -norm of a Calderón-Zygmund operator on \mathbb{R}^n with regular vector-valued kernel.*

Proof. Assume that $m - 1 \leq \alpha < m$ for some positive integer m . For any $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$, we have

$$g^q_{\alpha}(f)(x) = \|t^{\alpha} \partial_t^{\alpha} \mathcal{P}_t f(x)\|_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})} = \left\| \int_{\mathbb{R}^n} K_t(x-y) f(y) dy \right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})},$$

with

$$K_t(x-y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}} \Gamma(m-\alpha)} t^{\alpha} \int_0^{\infty} \partial_t^m \left(\frac{t+s}{((t+s)^2 + |x-y|^2)^{\frac{n+1}{2}}} \right) s^{m-\alpha-1} ds, \quad x, y \in \mathbb{R}^n, x \neq y, t > 0.$$

It can be proved that

$$\|K_t(x-y)\|_{\mathcal{L}(\mathbb{B}, L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t}))} \leq C \frac{1}{|x-y|^n},$$

and

$$\|\nabla_y K_t(x-y)\|_{\mathcal{L}(\mathbb{B}, L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t}))} + \|\nabla_x K_t(x-y)\|_{\mathcal{L}(\mathbb{B}, L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t}))} \leq C \frac{1}{|x-y|^{n+1}},$$

for any $x, y \in \mathbb{R}^n, x \neq y$. We leave the details of the proof to the reader. A sketch of it can be found in [2]. \square

Proposition 3.4. *Let \mathbb{B} be a Banach space which is of Lusin cotype q , $2 \leq q < \infty$. Then for every symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t \geq 0}$ with subordinated semigroup $\{\mathcal{P}_t\}_{t \geq 0}$ and for every (or, equivalently, for some) $p \in (1, \infty)$, there is a constant C such that*

$$(3.3) \quad \|g^q_k(f)\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad k = 1, 2, \dots, \quad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

Moreover, for any $0 < \alpha < \infty$, if

$$(3.4) \quad \|g^q_{\alpha}(f)\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad \forall f \in L^p_{\mathbb{B}}(\Omega),$$

then we have

$$(3.5) \quad \|g^q_{k\alpha}(f)\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad k = 1, 2, \dots, \quad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

Proof. For the case $k = 1$, the inequality (3.3) have been proved in [7]. We only need prove the cases $k = 2, 3, \dots$. We can prove it by induction. Assume that the inequality (3.3) is true for some $1 \leq k \in \mathbb{Z}$. Let us prove that it is true for $k + 1$ also. Since the inequality (3.3) is true for k , we know that the following operator

$$T : L_{\mathbb{B}}^q(\mathbb{R}^n) \longrightarrow L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^q(\mathbb{R}^n),$$

$$Tf(x, t) = t^k \partial_t^k \mathcal{P}_t f(x), \quad \forall f \in L_{\mathbb{B}}^q(\mathbb{R}^n)$$

is bounded. By Fubini's theorem we know that the operator

$$\tilde{T} : L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^q(\mathbb{R}^n) \longrightarrow L_{L_{\mathbb{B}}^q(\frac{ds}{s}, \frac{dt}{t})}^q(\mathbb{R}^n),$$

$$\tilde{T}F(x, s, t) = s \partial_s \mathcal{P}_s(F)(x, t), \quad \forall F(x, t) \in L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^q(\mathbb{R}^n)$$

is also bounded. Since \tilde{T} can be expressed as a Calderón–Zygmund operator with regular vector-valued kernel, by Remark 3.2 we get that $\tilde{T} : L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^p(\mathbb{R}^n) \longrightarrow L_{L_{\mathbb{B}}^q(\frac{ds}{s}, \frac{dt}{t})}^p(\mathbb{R}^n)$ is bounded for any $1 < p < \infty$. Hence, by Theorem 5.2 of [7], we know that $L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{ds}{s})$ is of Lusin cotype q .

Now given a symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t \geq 0}$ with subordinated semigroup $\{\mathcal{P}_t\}_{t \geq 0}$. As \mathbb{B} is of Lusin cotype q and $L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{ds}{s})$ also is of Lusin cotype q , we get that T is bounded from $L_{\mathbb{B}}^p(\Omega)$ to $L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^p(\Omega)$ and \tilde{T} is bounded from $L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^p(\Omega)$ to $L_{L_{\mathbb{B}}^q(\frac{ds}{s}, \frac{dt}{t})}^p(\Omega)$, for any $1 < p < \infty$. So the operator $\tilde{T} \circ T$ is bounded from $L_{\mathbb{B}}^p(\Omega)$ to $L_{L_{\mathbb{B}}^q(\frac{ds}{s}, \frac{dt}{t})}^p(\Omega)$, for any $1 < p < \infty$, and by (2.7) we have

$$\begin{aligned} \tilde{T} \circ Tf(x, t, s) &= \tilde{T}(Tf(x, t))(s) = \tilde{T}(t^k \partial_t^k \mathcal{P}_t f(x))(s) \\ (3.6) \quad &= s \partial_s \mathcal{P}_s(t^k \partial_t^k \mathcal{P}_t f)(x) = st^k \partial_s \partial_t^k \mathcal{P}_s \mathcal{P}_t f(x) \\ &= st^k \partial_s \partial_t^k \mathcal{P}_{s+t} f(x) = st^k \partial_u^{k+1} \mathcal{P}_u f|_{u=t+s}(x). \end{aligned}$$

So there exists a constant C such that

$$\begin{aligned} \|f\|_{L_{\mathbb{B}}^p(\Omega)}^p &\geq C \left\| \tilde{T} \circ Tf \right\|_{L_{L_{\mathbb{B}}^q(\frac{ds}{s}, \frac{dt}{t})}^p(\Omega)}^p \\ &= C \left\| st^k \partial_u^{k+1} \mathcal{P}_u f|_{u=t+s}(x) \right\|_{L_{L_{\mathbb{B}}^q(\frac{ds}{s}, \frac{dt}{t})}^p(\Omega)}^p \\ &= C \left\| \left(\int_0^\infty \int_0^\infty \left\| st^k \partial_u^{k+1} \mathcal{P}_u f|_{u=t+s}(x) \right\|_{\mathbb{B}}^q \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)}^p \\ (3.7) \quad &= C \left\| \left(\int_0^\infty \int_t^\infty t^{kq} (s-t)^q \left\| \partial_s^{k+1} \mathcal{P}_s f \right\|_{\mathbb{B}}^q \frac{ds}{s-t} \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)}^p \\ &= C \left\| \left(\int_0^\infty \left\| \partial_s^{k+1} \mathcal{P}_s f \right\|_{\mathbb{B}}^q \int_0^s t^{kq-1} (s-t)^{q-1} dt ds \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)}^p \\ &= C(\mathbf{B}(kq, q))^{\frac{2}{q}} \left\| \left(\int_0^\infty s^{(k+1)q} \left\| \partial_s^{k+1} \mathcal{P}_s f \right\|_{\mathbb{B}}^q \frac{ds}{s} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)}^p \\ &= C(\mathbf{B}(kq, q))^{\frac{2}{q}} \|g_{k+1}^q(f)\|_{L^p(\Omega)}^p. \end{aligned}$$

Whence

$$\|g_{k+1}^q(f)\|_{L^p(\Omega)} \leq C \|f\|_{L_{\mathbb{B}}^p(\Omega)}, \quad \forall f \in L_{\mathbb{B}}^p(\Omega).$$

Then we get the inequality (3.3) for any $k \in \mathbb{Z}_+$.

We can prove inequality (3.5) under the assumption (3.4) with the similar argument as above. The only difference is that we should define T by

$$Tf(x, t) = t^{k\alpha} \partial_t^{k\alpha} \mathcal{P}_t f(x), \quad \forall f \in L_{\mathbb{B}}^q(\mathbb{R}^n),$$

and define \tilde{T} by

$$\tilde{T}F(x, s, t) = s^\alpha \partial_s^\alpha \mathcal{P}_s F(x, t), \quad \forall F(x, t) \in L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^q(\mathbb{R}^n).$$

And by Proposition 3.3 we know that in this case \tilde{T} can be expressed as a Calderón–Zygmund operator also. \square

The following theorem is proved in [7] which we will use later.

Theorem 3.5. [7, Theorem 3.2] *Let \mathbb{B} be a Banach space and $1 < p, q < \infty$. Let $h(x, t)$ be a function in $L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^p(\Omega)$. Consider the operator Q defined by $Qh(x) = \int_0^\infty \partial_t \mathcal{P}_t h(x, t) dt$, $x \in \Omega$. Then for nice function h we have*

$$\|g_1^q(Qh)\|_{L^p(\Omega)} \leq C_{p,q} \|h\|_{L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^p(\Omega)},$$

where the constant $C_{p,q}$ depends only on p and q .

4. PROOFS OF THEOREM A AND THEOREM B

Now we are in a position to prove Theorem A and Theorem B.

Proof of Theorem A. (i) \Rightarrow (ii). Since \mathbb{B} is of Lusin cotype q , by Proposition 3.4 we have

$$\|g_k^q(f)\|_{L^p(\Omega)} \leq C \|f\|_{L_{\mathbb{B}}^p(\Omega)}, \quad k = 1, 2, \dots, \quad \forall f \in L_{\mathbb{B}}^p(\Omega).$$

Then, for any $\alpha > 0$, there exists $k \in \mathbb{N}$ such that $\alpha < k$. By Proposition 3.1, we have

$$\|g_\alpha^q(f)\|_{L^p(\Omega)} \leq C \|g_k^q(f)\|_{L^p(\Omega)} \leq C \|f\|_{L_{\mathbb{B}}^p(\Omega)}, \quad \forall f \in L_{\mathbb{B}}^p(\Omega).$$

(ii) \Rightarrow (i). Since $\|g_\alpha^q(f)\|_{L^p(\Omega)} \leq C \|f\|_{L_{\mathbb{B}}^p(\Omega)}$ for any $f \in L_{\mathbb{B}}^p(\Omega)$, by Proposition 3.4 there exists an integer k such that $k\alpha > 1$ and

$$\|g_{k\alpha}^q(f)\|_{L^p(\Omega)} \leq C \|f\|_{L_{\mathbb{B}}^p(\Omega)}$$

for any $f \in L_{\mathbb{B}}^p(\Omega)$. By Proposition 3.1, we have

$$\|g_1^q(f)\|_{L^p(\Omega)} \leq C \|g_{k\alpha}^q(f)\|_{L^p(\Omega)} \leq C \|f\|_{L_{\mathbb{B}}^p(\Omega)}$$

for any $f \in L_{\mathbb{B}}^p(\Omega)$. Hence, by Theorem 1.1, \mathbb{B} is of Lusin cotype q . \square

Proof of Theorem B. (i) \Rightarrow (ii). It is easy to deduce from (2.15) that for any $f, g \in L^2(\Omega)$

$$(4.1) \quad \int_{\Omega} (f - E_0(f))(g - E_0(g)) d\mu = \frac{4^\alpha}{\Gamma(2\alpha)} \int_{\Omega} \int_0^\infty (t^\alpha \partial_t^\alpha \mathcal{P}_t f)(t^\alpha \partial_t^\alpha \mathcal{P}_t g) \frac{dt}{t} d\mu.$$

Now we use duality. Fix two functions $f \in L_{\mathbb{B}}^p(\Omega)$ and $g \in L_{\mathbb{B}^*}^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Without loss of generality, we may assume that f and g are in the algebraic tensor products $(L^p(\Omega) \cap L^2(\Omega)) \otimes \mathbb{B}$ and $(L^{p'}(\Omega) \cap L^2(\Omega)) \otimes \mathbb{B}^*$, respectively. With $\langle \cdot, \cdot \rangle$ denoting the duality between \mathbb{B} and \mathbb{B}^* , we have

$$(4.2) \quad \int_{\Omega} \langle f, g \rangle d\mu = \int_{\Omega} \langle E_0(f), E_0(g) \rangle d\mu + \int_{\Omega} \langle f - E_0(f), g - E_0(g) \rangle d\mu.$$

The first term on the right is easy to be estimated:

$$(4.3) \quad \left| \int_{\Omega} \langle E_0(f), E_0(g) \rangle d\mu \right| \leq \|E_0(f)\|_{L_{\mathbb{B}}^p(\Omega)} \|E_0(g)\|_{L_{\mathbb{B}^*}^{p'}(\Omega)} \leq \|E_0(f)\|_{L_{\mathbb{B}}^p(\Omega)} \|g\|_{L_{\mathbb{B}^*}^{p'}(\Omega)}.$$

For the second one, by (4.1) and Hölder's inequality

$$(4.4) \quad \begin{aligned} \left| \int_{\Omega} \langle f - E_0(f), g - E_0(g) \rangle d\mu \right| &= \frac{4^\alpha}{\Gamma(2\alpha)} \left| \int_{\Omega} \int_0^\infty \langle t^\alpha \partial_t^\alpha \mathcal{P}_t f, t^\alpha \partial_t^\alpha \mathcal{P}_t g \rangle \frac{dt}{t} d\mu \right| \\ &\leq \frac{4^\alpha}{\Gamma(2\alpha)} \int_{\Omega} \int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f\|_{\mathbb{B}} \|t^\alpha \partial_t^\alpha \mathcal{P}_t g\|_{\mathbb{B}^*} \frac{dt}{t} d\mu \\ &\leq \frac{4^\alpha}{\Gamma(2\alpha)} \|g_\alpha^q(f)\|_{L^p(\Omega)} \|g_\alpha^{q'}(g)\|_{L^{p'}(\Omega)}. \end{aligned}$$

Now since \mathbb{B} is of Lusin type q , \mathbb{B}^* is of Lusin cotype q' . Thus by Theorem A,

$$(4.5) \quad \|g_\alpha^{q'}(g)\|_{L^{p'}(\Omega)} \leq C \|g\|_{L^{p'}(\Omega)}.$$

Combining (4.2)–(4.5), we get

$$\left| \int_\Omega \langle f, g \rangle d\mu \right| \leq \left(\|E_0(f)\|_{L^p(\Omega)} + C \|g_\alpha^q(f)\|_{L^p(\Omega)} \right) \|g\|_{L^{p'}(\Omega)},$$

which gives (ii) by taking the supremum over all g as above such that $\|g\|_{L^{p'}(\Omega)} \leq 1$.

(ii) \Rightarrow (i). We only need consider the particular case on \mathbb{R}^n . In this case, $E_0(f) = 0$ for any $f \in L^p_{\mathbb{B}}(\mathbb{R}^n)$. Assuming $p = q$ and $k - 1 \leq \alpha < k$ for some $k \in \mathbb{Z}_+$, by Proposition 3.1 we have

$$(4.6) \quad \|f\|_{L^q_{\mathbb{B}}(\mathbb{R}^n)} \leq C \|g_\alpha^q(f)\|_{L^q(\mathbb{R}^n)} \leq C \|g_k^q(f)\|_{L^q(\mathbb{R}^n)},$$

for any $f \in L^q_{\mathbb{B}}(\mathbb{R}^n)$. By using (3.7) and (3.6), we have

$$\left(\int_0^\infty s^{kq} \|\partial_s^k \mathcal{P}_s f\|_{\mathbb{B}}^q \frac{ds}{s} \right)^{\frac{1}{q}} = C \left(\int_0^\infty \int_0^\infty s_1^q s_2^{(k-1)q} \|\partial_{s_2}^{k-1} \mathcal{P}_{s_2} (\partial_{s_1} \mathcal{P}_{s_1}) f\|_{\mathbb{B}}^q \frac{ds_2}{s_2} \frac{ds_1}{s_1} \right)^{\frac{1}{q}}.$$

By iterating the argument, we can get

$$\left(\int_0^\infty s^{kq} \|\partial_s^k \mathcal{P}_s f\|_{\mathbb{B}}^q \frac{ds}{s} \right)^{\frac{1}{q}} = C \left(\int_0^\infty \cdots \int_0^\infty s_1^q \cdots s_k^q \|\partial_{s_1} \mathcal{P}_{s_1} \cdots \partial_{s_k} \mathcal{P}_{s_k} f\|_{\mathbb{B}}^q \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} \right)^{\frac{1}{q}}.$$

Therefore we can choose a function $b(x, s_1, \dots, s_k) \in L^q_{L_{\mathbb{B}}\left(\frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k}\right)}(\mathbb{R}^n)$ of unit norm such that

$$\|g_k^{q'}(f)\|_{L^{q'}(\mathbb{R}^n)} = C \int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \langle s_1 \cdots s_k \partial_{s_1} \mathcal{P}_{s_1} \cdots \partial_{s_k} \mathcal{P}_{s_k} f(x), b(x, s_1, \dots, s_k) \rangle \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} dx.$$

We may assume that f and b are nice enough to legitimate the calculations below. By Fubini's theorem, Hölder's inequality and (4.6), we have

$$\begin{aligned} & \|g_k^{q'}(f)\|_{L^{q'}(\mathbb{R}^n)} \\ &= C \int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \langle s_1 \cdots s_k \partial_{s_1} \mathcal{P}_{s_1} \cdots \partial_{s_k} \mathcal{P}_{s_k} f(x), b(x, s_1, \dots, s_k) \rangle \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} dx \\ &= C \int_{\mathbb{R}^n} \left\langle f(x), \int_0^\infty \cdots \int_0^\infty s_1 \cdots s_k \partial_{s_1} \mathcal{P}_{s_1} \cdots \partial_{s_k} \mathcal{P}_{s_k} b(x, s_1, \dots, s_k) \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} \right\rangle dx \\ (4.7) \quad & \leq C \|f\|_{L^{q'}_{\mathbb{B}^*}(\mathbb{R}^n)} \left\| \int_0^\infty \cdots \int_0^\infty s_1 \cdots s_k \partial_{s_1} \mathcal{P}_{s_1} \cdots \partial_{s_k} \mathcal{P}_{s_k} b(x, s_1, \dots, s_k) \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} \right\|_{L^q_{\mathbb{B}}(\mathbb{R}^n)} \\ & \leq C \|f\|_{L^{q'}_{\mathbb{B}^*}(\mathbb{R}^n)} \left\| g_k^q \left(\int_0^\infty \cdots \int_0^\infty s_1 \cdots s_k \partial_{s_1} \mathcal{P}_{s_1} \cdots \partial_{s_k} \mathcal{P}_{s_k} b(x, s_1, \dots, s_k) \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} \right) \right\|_{L^q(\mathbb{R}^n)} \\ & =: C \|f\|_{L^{q'}_{\mathbb{B}^*}(\mathbb{R}^n)} \|g_k^q(G_k(b))\|_{L^q(\mathbb{R}^n)}, \end{aligned}$$

where

$$G_k(b) = \int_0^\infty \cdots \int_0^\infty s_1 \cdots s_k \partial_{s_1} \mathcal{P}_{s_1} \cdots \partial_{s_k} \mathcal{P}_{s_k} b(x, s_1, \dots, s_k) \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k}, \quad k \in \mathbb{Z}_+.$$

Using (3.7), Fubini's theorem and Theorem 3.5 repeatedly, we have

(4.8)

$$\begin{aligned}
& \|g_k^q(G_k(b))\|_{L^q(\mathbb{R}^n)}^q \\
& \leq C \int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \left\| t_1 \partial_{t_1} \mathcal{P}_{t_1} \cdots t_k \partial_{t_k} \mathcal{P}_{t_k} (G_k(b)) \right\|_{\mathbb{B}}^q \frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k} dx \\
& = C \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \left\| t_k \partial_{t_k} \mathcal{P}_{t_k} \left[\int_0^\infty s_k \partial_{s_k} \mathcal{P}_{s_k} (t_1 \partial_{t_1} \mathcal{P}_{t_1} \cdots t_{k-1} \partial_{t_{k-1}} \mathcal{P}_{t_{k-1}} G_{k-1}(b)) \frac{ds_k}{s_k} \right] \right\|_{\mathbb{B}}^q \\
& \quad \frac{dt_k}{t_k} dx \frac{dt_1}{t_1} \cdots \frac{dt_{k-1}}{t_{k-1}} \\
& \leq C \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \left\| t_1 \cdots t_{k-1} \partial_{t_1} \mathcal{P}_{t_1} \cdots \partial_{t_{k-1}} \mathcal{P}_{t_{k-1}} G_{k-1}(b) \right\|_{\mathbb{B}}^q \frac{ds_k}{s_k} dx \frac{dt_1}{t_1} \cdots \frac{dt_{k-1}}{t_{k-1}} \\
& \vdots \\
& \leq C \int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \|b(x, s_1, \dots, s_k)\|_{\mathbb{B}}^q \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} dx = C.
\end{aligned}$$

Combining (4.7) and (4.8), we get

$$\left\| g_k^{q'}(f) \right\|_{L^{q'}(\mathbb{R}^n)} \leq C \|f\|_{L_{\mathbb{B}^*}^{q'}(\mathbb{R}^n)}.$$

By Theorem A, \mathbb{B}^* is of Lusin cotype q' . Hence \mathbb{B} is of Lusin type q .

If $p \neq q$, it suffices to prove that the operator $b \rightarrow g_k^q(G_k(b))$ maps $L_{L_{\mathbb{B}}^q\left(\frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k}\right)}^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

To that end we shall use the theory of vector-valued Calderón-Zygmund operators. We borrow this idea from [8]. Let us consider the operator

$$T(b)(x, t_1, \dots, t_k) = t_1 \partial_{t_1} \mathcal{P}_{t_1} \cdots t_k \partial_{t_k} \mathcal{P}_{t_k} \int_0^\infty \cdots \int_0^\infty s_1 \partial_{s_1} \mathcal{P}_{s_1} \cdots s_k \partial_{s_k} \mathcal{P}_{s_k} b(x, s_1, \dots, s_k) \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k}.$$

Clearly,

$$\|T(b)(x, t_1, \dots, t_k)\|_{L_{L_{\mathbb{B}}^q\left(\frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k}\right)}^p(\mathbb{R}^n)} = \|g_k^q(G_k(b))\|_{L^p(\mathbb{R}^n)}.$$

Therefore it is enough to prove

$$T : L_{L_{\mathbb{B}}^q\left(\frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k}\right)}^p(\mathbb{R}^n) \longrightarrow L_{L_{\mathbb{B}}^q\left(\frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k}\right)}^p(\mathbb{R}^n).$$

Hence as we already know that T is bounded in the case $p = q$, in order to get the case $p \neq q$ it suffices to show that the kernel of T satisfies the standard estimates, see Remark 3.2. For simply and essentially, we only need consider the case when $k = 2$. So

$$\begin{aligned}
T(b)(x, t_1, t_2) &= t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} \int_0^\infty \int_0^\infty s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2} (b)(x, s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2} \\
&= \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2} (x - y) b(y, s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2} dy.
\end{aligned}$$

Then the operator-valued kernel $K(x)$ is $\int_0^\infty \int_0^\infty t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2}(x) \frac{ds_1}{s_1} \frac{ds_2}{s_2}$. For any $b(s_1, s_2) \in L_{\mathbb{B}}^q\left(\frac{ds_1}{s_1} \frac{ds_2}{s_2}\right)$ with unit norm, we have

$$\begin{aligned}
\|K(x)b\|_{\mathbb{B}} &= \left\| \int_0^\infty \int_0^\infty t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2}(x) b(s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2} \right\|_{\mathbb{B}} \\
&= \left\| \int_0^\infty \int_0^\infty t_1 t_2 s_1 s_2 \partial_u^4 \mathcal{P}_u(x) \Big|_{u=t_1+t_2+s_1+s_2} b(s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2} \right\|_{\mathbb{B}} \\
&\leq \int_0^\infty \int_0^\infty t_1 t_2 s_1 s_2 \partial_u^4 \mathcal{P}_u(x) \Big|_{u=t_1+t_2+s_1+s_2} \|b(s_1, s_2)\|_{\mathbb{B}} \frac{ds_1}{s_1} \frac{ds_2}{s_2} \\
&\leq \|b\|_{L_{\mathbb{B}}^q\left(\frac{ds_1}{s_1} \frac{ds_2}{s_2}\right)} \left\{ \int_0^\infty \int_0^\infty \left(t_1 t_2 s_1 s_2 \partial_u^4 \mathcal{P}_u(x) \Big|_{u=t_1+t_2+s_1+s_2} \right)^{q'} \frac{ds_1}{s_1} \frac{ds_2}{s_2} \right\}^{\frac{1}{q'}} \\
&\leq C \left\{ \int_0^\infty \int_0^\infty \left(\frac{t_1 t_2 s_1 s_2}{(t_1 + t_2 + s_1 + s_2 + |x|)^{n+4}} \right)^{q'} \frac{ds_1}{s_1} \frac{ds_2}{s_2} \right\}^{\frac{1}{q'}} \\
&\leq C \frac{t_1 t_2}{(t_1 + t_2 + |x|)^{n+2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|K(x)b\|_{L_{\mathbb{B}}^q\left(\frac{dt_1}{t_1} \frac{dt_2}{t_2}\right)} &= \left(\int_0^\infty \int_0^\infty \|K(x)b\|_{\mathbb{B}}^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{\frac{1}{q}} \\
&\leq C \left(\int_0^\infty \int_0^\infty \left(\frac{t_1 t_2}{(t_1 + t_2 + |x|)^{n+2}} \right)^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{\frac{1}{q}} \leq \frac{C}{|x|^n}.
\end{aligned}$$

Similarly, we can show that

$$\|\nabla K(x)\| \leq \frac{C}{|x|^{n+1}}.$$

Therefore, K is a regular vector-valued kernel and the proof is finished. \square

5. POISSON SEMIGROUP ON \mathbb{R}^n

In this section, we devote to study the fractional area function and the fractional g_λ^* -function on \mathbb{R}^n in the vector-valued case. Our main goal is to prove the analogous results with Theorem A and Theorem B related to these two functions on \mathbb{R}^n .

Let \mathbb{B} be a Banach space, $0 < \alpha < \infty$, $\lambda > 1$, and $1 < q < \infty$. We define the fractional area function on \mathbb{R}^n as

$$S_\alpha^q(f)(x) = \left(\iint_{\Gamma(x)} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{q}}, \quad \forall f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\mathbb{R}^n),$$

where $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$, and define the fractional g_λ^* -function on \mathbb{R}^n as

$$g_{\lambda, \alpha}^{q,*}(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x - y| + t} \right)^{\lambda n} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{q}}, \quad \forall f \in \bigcup_{1 \leq p \leq \infty} L_{\mathbb{B}}^p(\mathbb{R}^n).$$

The following proposition demonstrate that the vector-valued fractional area function S_α^q can be treated as an $L_{\mathbb{B}}^q(\Gamma(0), \frac{dy dt}{t^{n+1}})$ -norm of a Calderón–Zygmund operator.

Proposition 5.1. *Given a Banach space \mathbb{B} , $1 < q < \infty$ and $0 < \alpha < \infty$, then $S_\alpha^q(f)$ can be expressed as an $L_{\mathbb{B}}^q(\Gamma(0), \frac{dy dt}{t^{n+1}})$ -norm of a Calderón–Zygmund operator on \mathbb{R}^n with regular vector-valued kernel.*

Proof. Assume that $m - 1 \leq \alpha < m$ for some positive integer m . For any $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$, by changing of variables we have

$$S_\alpha^q(f)(x) = \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(x+y)\|_{L_\mathbb{B}^q(\Gamma(0), \frac{dydt}{t^{n+1}})} = \left\| \int_{\mathbb{R}^n} K_{y,t}(x,z) f(z) dz \right\|_{L_\mathbb{B}^q(\Gamma(0), \frac{dydt}{t^{n+1}})},$$

where

$$K_{y,t}(x,z) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}} \Gamma(m-\alpha)} t^\alpha \int_0^\infty \partial_t^m \left(\frac{t+s}{((t+s)^2 + |x+y-z|^2)^{\frac{n+1}{2}}} \right) s^{m-\alpha-1} ds, \quad x, y, z \in \mathbb{R}^n, t > 0.$$

It can be proved that

$$\|K_{y,t}(x,z)\|_{\mathcal{L}(\mathbb{B}, L_\mathbb{B}^q(\Gamma(0), \frac{dydt}{t^{n+1}}))} \leq C \frac{1}{|x-z|^n},$$

and

$$\|\nabla_x K_{y,t}(x,z)\|_{\mathcal{L}(\mathbb{B}, L_\mathbb{B}^q(\Gamma(0), \frac{dydt}{t^{n+1}}))} + \|\nabla_z K_{y,t}(x,z)\|_{\mathcal{L}(\mathbb{B}, L_\mathbb{B}^q(\Gamma(0), \frac{dydt}{t^{n+1}}))} \leq C \frac{1}{|x-z|^{n+1}},$$

for any $x, z \in \mathbb{R}^n, x \neq z$. We leave the details of the proof to the reader. \square

Together with Proposition 3.3, Proposition 5.1 and Remark 3.2, we can immediately get the following theorem for g_α^q and S_α^q with $1 < q < \infty$ and $0 < \alpha < \infty$.

Theorem 5.2. *Given a Banach space \mathbb{B} , $1 < q < \infty$ and $0 < \alpha < \infty$, let U be either the fractional Littlewood-Paley g -function g_α^q or the fractional area function S_α^q , then the following statements are equivalent:*

- (i) U maps $L_{c,\mathbb{B}}^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.
- (ii) U maps $H_\mathbb{B}^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.
- (iii) U maps $L_\mathbb{B}^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for any (or, equivalently, for some) $p \in (1, \infty)$.
- (iv) U maps $BMO_{c,\mathbb{B}}(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.
- (v) U maps $L_\mathbb{B}^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Theorem 5.3. *Given a Banach space \mathbb{B} and $2 \leq q < \infty$, the following statements are equivalent:*

- (i) \mathbb{B} is of Lusin cotype q .
- (ii) For every (or, equivalently, for some) positive integer n , for every (or, equivalently, for some) $p \in (1, \infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant $C > 0$ such that

$$\|S_\alpha^q(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L_\mathbb{B}^p(\mathbb{R}^n)}, \quad \forall f \in L_\mathbb{B}^p(\mathbb{R}^n).$$

Proof. (i) \Rightarrow (ii). By Fubini's theorem, we have

$$\begin{aligned} (5.1) \quad \|S_\alpha^q(f)\|_{L^q(\mathbb{R}^n)}^q &= \int_{\mathbb{R}^n} \int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \left(\int_{\mathbb{R}^n} \chi_{|x-y|<t} dx \right) \frac{dydt}{t^{n+1}} \\ &= \int_{\mathbb{R}^n} \int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dydt}{t} = \|g_\alpha^q(f)\|_{L^q(\mathbb{R}^n)}^q. \end{aligned}$$

Since \mathbb{B} is of Lusin cotype q , by (5.1) and Theorem A we get

$$\|S_\alpha^q(f)\|_{L^q(\mathbb{R}^n)} = \|g_\alpha^q(f)\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L_\mathbb{B}^q(\mathbb{R}^n)}, \quad \forall f \in L_\mathbb{B}^q(\mathbb{R}^n).$$

Hence, by Theorem 5.2

$$\|S_\alpha^q(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L_\mathbb{B}^p(\mathbb{R}^n)}, \quad \forall f \in L_\mathbb{B}^p(\mathbb{R}^n), 1 < p < \infty.$$

(ii) \Rightarrow (i). We only need prove that there exists a constant C such that

$$(5.2) \quad g_\alpha^q(f)(x) \leq C S_\alpha^q(f)(x), \quad \forall x \in \mathbb{R}^n,$$

for a big enough class of nice functions in $L_\mathbb{B}^p(\mathbb{R}^n)$. Then we have $\|g_\alpha^q(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L_\mathbb{B}^p(\mathbb{R}^n)}$. By Theorem A, \mathbb{B} is of Lusin cotype q .

Now, let us prove (5.2). We shall follow those ideas in [13]. It suffices to prove it for $x = 0$. Let us denote by $B(0,t)$ the ball in \mathbb{R}^{n+1} centered at $(0,t)$ and tangent to the boundary of the cone

$\Gamma(0)$. Then the radius of $B(0, t)$ is $\frac{\sqrt{2}}{2}t$. Now the partial derivative $\partial_t^\alpha \mathcal{P}_t f(x)$ is, like $\mathcal{P}_t f(x)$, harmonic function. Thus by the mean-value theorem, we have

$$\partial_t^\alpha \mathcal{P}_t f(0) = \frac{1}{|B(0, t)|} \iint_{B(0, t)} \partial_s^\alpha \mathcal{P}_s f(x) dx ds.$$

By Hölder's inequality,

$$\|\partial_t^\alpha \mathcal{P}_t f(0)\|_{\mathbb{B}} \leq \frac{1}{|B(0, t)|} \iint_{B(0, t)} \|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}} dx ds \leq \frac{1}{|B(0, t)|^{\frac{1}{q}}} \left(\iint_{B(0, t)} \|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q dx ds \right)^{\frac{1}{q}}.$$

Integrating this inequality, we obtain

$$\begin{aligned} \int_0^\infty t^{\alpha q} \|\partial_t^\alpha \mathcal{P}_t f(0)\|_{\mathbb{B}}^q \frac{dt}{t} &\leq C \int_0^\infty t^{\alpha q - n - 2} \iint_{B(0, t)} \|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q dx ds dt \\ &\leq C \iint_{\Gamma(0)} \left(\int_{c_1 s}^{c_2 s} t^{\alpha q - n - 2} dt \right) \|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q dx ds \leq C \iint_{\Gamma(0)} \|s^\alpha \partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q \frac{dx ds}{s^{n+1}} \end{aligned}$$

by using Fubini's theorem and $(x, s) \in B(0, t)$ implying $c_1 s \leq t \leq c_2 s$, for two positive constants c_1 and c_2 . Hence, we get inequality (5.2). \square

Theorem 5.4. *Given a Banach space \mathbb{B} and $1 < q \leq 2$, the following statements are equivalent:*

- (i) \mathbb{B} is of Lusin type q .
- (ii) For every (or, equivalently, for some) positive integer n , for every (or, equivalently, for some) $p \in (1, \infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant $C > 0$ such that

$$\|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C \|S_\alpha^q(f)\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L_{\mathbb{B}}^p(\mathbb{R}^n).$$

Proof. (i) \Rightarrow (ii). Since \mathbb{B} is of Lusin type q , by Theorem B and (5.2) we have

$$\|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C \|g_\alpha^q(f)\|_{L^p(\mathbb{R}^n)} \leq C \|S_\alpha^q(f)\|_{L^p(\mathbb{R}^n)}.$$

(ii) \Rightarrow (i). We shall prove $\|S_\alpha^{q'}(g)\|_{L^{p'}(\mathbb{R}^n)} \leq C \|g\|_{L_{\mathbb{B}^*}^{p'}(\mathbb{R}^n)}$. We can choose $b \in L_{L_{\mathbb{B}}^q(\Gamma(0), \frac{dz dt}{t^{n+1}})}^q(\mathbb{R}^n)$ of unit norm such that

$$\begin{aligned} \|S_\alpha^{q'}(g)\|_{L^{p'}(\mathbb{R}^n)} &= \|t^\alpha \partial_t^\alpha \mathcal{P}_t g(y - z)\|_{L_{L_{\mathbb{B}^*}^{q'}(\Gamma(0), \frac{dz dt}{t^{n+1}})}^{q'}(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} \int_{\Gamma(0)} \langle t^\alpha \partial_t^\alpha \mathcal{P}_t g(y - z), b(y, z, t) \rangle \frac{dz dt}{t^{n+1}} dy \\ &= \int_{\mathbb{R}^n} \int_{\Gamma(0)} \left\langle \int_{\mathbb{R}^n} t^\alpha \partial_t^\alpha \mathcal{P}_t (y - z - \tilde{z}) g(\tilde{z}) d\tilde{z}, b(y, z, t) \right\rangle \frac{dz dt}{t^{n+1}} dy \\ &= \int_{\mathbb{R}^n} \left\langle g(\tilde{z}), \int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial_t^\alpha \mathcal{P}_t (y - z - \tilde{z}) b(y, z, t) dy \frac{dz dt}{t^{n+1}} \right\rangle d\tilde{z} \\ &\leq \|g\|_{L_{\mathbb{B}^*}^{p'}(\mathbb{R}^n)} \|G(b)\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq \|g\|_{L_{\mathbb{B}^*}^{p'}(\mathbb{R}^n)} \|S_\alpha^q(G(b))\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)}, \end{aligned}$$

where $G(b)(\tilde{z}) = \int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial_t^\alpha \mathcal{P}_t (y - z - \tilde{z}) b(y, z, t) dy \frac{dz dt}{t^{n+1}}$ and in the last inequality we used the hypothesis. Let us observe that we will have proved the result as soon as we prove $\|S_\alpha^q(G(b))\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C \|b\|_{L_{L_{\mathbb{B}}^q(\Gamma(0), \frac{dz dt}{t^{n+1}})}^q(\mathbb{R}^n)}$. We shall prove this by following a parallel argument to the proof of (ii) \Rightarrow (i)

in Theorem B and we also borrow the ideal from [8]. Observe that

$$\begin{aligned}
& S_\alpha^q(G(b))(x) \\
&= \left(\int_{\Gamma(0)} \left\| s^\alpha \partial_s^\alpha \mathcal{P}_s \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial_t^\alpha \mathcal{P}_t(y-z-\cdot) b(y, z, t) dy \frac{dzdt}{t^{n+1}} \right) (x+u) \right\|_{\mathbb{B}}^q \frac{duds}{s^{n+1}} \right)^{1/q} \\
&= \left(\int_{\Gamma(0)} \left\| s^\alpha \partial_s^\alpha \mathcal{P}_s \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial_t^\alpha \mathcal{P}_t(-y+z+\cdot) b(y, z, t) dy \frac{dzdt}{t^{n+1}} \right) (x+u) \right\|_{\mathbb{B}}^q \frac{duds}{s^{n+1}} \right)^{1/q} \\
&= \left(\int_{\Gamma(0)} \left\| \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} s^\alpha \partial_s^\alpha \mathcal{P}_s t^\alpha \partial_t^\alpha \mathcal{P}_t(-y+z+x+u) b(y, z, t) dy \frac{dzdt}{t^{n+1}} \right) \right\|_{\mathbb{B}}^q \frac{duds}{s^{n+1}} \right)^{1/q} \\
&= \left(\int_{\Gamma(0)} \left\| \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} s^\alpha t^\alpha \partial_u^{2\alpha} \mathcal{P}_u|_{u=s+t}(-y+z+x+u) b(y, z, t) dy \frac{dzdt}{t^{n+1}} \right) \right\|_{\mathbb{B}}^q \frac{duds}{s^{n+1}} \right)^{1/q}.
\end{aligned}$$

It is an easy exercise to prove that

$$|s^\alpha t^\alpha \partial_u^{2\alpha} \mathcal{P}_u|_{u=s+t}| \leq C \frac{s^\alpha t^\alpha}{(s+t+|x|)^{n+2\alpha}}.$$

In this circumstances, it can be proved that the operator

$$b \longrightarrow \mathcal{U}(b)(x, u, s) = \int_{\mathbb{R}^n} \int_{\Gamma(0)} s^\alpha t^\alpha \partial_u^{2\alpha} \mathcal{P}_u|_{u=s+t}(-y+z+x+u) b(y, z, t) \frac{dzdt}{t^{n+1}} dy$$

can be handled by using Calderón–Zygmund techniques and \mathcal{U} is bounded on $L_{L_{\mathbb{B}}^q(\Gamma(0), \frac{duds}{s^{n+1}})}^p(\mathbb{R}^n)$ for every $1 < p, q < \infty$ and every Banach space \mathbb{B} , see the details in [8, Section 2]. The proof of the theorem ends by observing that $S_\alpha^q(G(b)) = \|\mathcal{U}(b)\|_{L_{\mathbb{B}}^q(\Gamma(0), \frac{duds}{s^{n+1}})}$. \square

Now, let us consider the relationship between the geometry properties of the Banach space \mathbb{B} and the fractional g_λ^* -function $g_{\lambda, \alpha}^{q, *}$.

Theorem 5.5. *Given a Banach space \mathbb{B} , $2 \leq q < \infty$ and $\lambda > 1$, the following statements are equivalent:*

- (i) \mathbb{B} is of *Lusin cotype* q .
- (ii) *For every (or, equivalently, for some) positive integer n , for every (or, equivalently, for some) $p \in [q, \infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant $C > 0$ such that*

$$\left\| g_{\lambda, \alpha}^{q, *}(f) \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)}, \quad \forall f \in L_{\mathbb{B}}^p(\mathbb{R}^n).$$

Proof. (i) \Rightarrow (ii). Since $\lambda > 1$, the function $(1 + |x|)^{-\lambda n}$ is integrable and hence for good enough function $h(x) \geq 0$, we have

$$(5.3) \quad \sup_{t>0} \int_{\mathbb{R}^n} \frac{1}{t^n} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} h(y) dy \leq CMh(x),$$

where Mh is the Hardy–Littlewood maximal function of h . By (5.3) and Hölder’s inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(g_{\lambda, \alpha}^{q, *}(f)(x) \right)^q h(x) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \frac{dt}{t^{n+1}} dy h(x) dx \\
&\leq C \int_{\mathbb{R}^n} (g_\alpha^q(f)(y))^q Mh(y) dy \leq C \|g_\alpha^q(f)\|_{L^p(\mathbb{R}^n)}^q \|Mh\|_{L^{\frac{p}{p-q}}(\mathbb{R}^n)}.
\end{aligned}$$

Here, when $p = q$, let $L^{\frac{p}{p-q}}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. Since M is bounded on $L^r(\mathbb{R}^n)$ ($1 < r \leq \infty$), we get

$$\int_{\mathbb{R}^n} \left(g_{\lambda, \alpha}^{q, *}(f)(x) \right)^q h(x) dx \leq C \|g_\alpha^q(f)\|_{L^p(\mathbb{R}^n)}^q \|h\|_{L^{\frac{p}{p-q}}(\mathbb{R}^n)}.$$

Taking supremum over all h in $L^{\frac{p}{p-q}}(\mathbb{R}^n)$, we get

$$(5.4) \quad \left\| g_{\lambda, \alpha}^{q, *}(f) \right\|_{L^p(\mathbb{R}^n)} \leq C \|g_\alpha^q(f)\|_{L^p(\mathbb{R}^n)}, \quad q \leq p.$$

Since \mathbb{B} is of Lusin cotype q , by Theorem A and (5.4) we get $\|g_{\lambda,\alpha}^{q,*}(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)}$.

(ii) \Rightarrow (i). On the domain $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^n : |y - x| < t\}$, we have

$$\left(\frac{t}{|x - y| + t}\right)^{\lambda n} > \left(\frac{1}{2}\right)^{\lambda n}.$$

Hence

$$\begin{aligned} S_{\alpha}^q(f)(x) &= \left(\iint_{\Gamma(x)} \|t^{\alpha} \partial_t^{\alpha} \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{q}} \\ &\leq \left(\iint_{\Gamma(x)} 2^{\lambda n} \left(\frac{t}{|x - y| + t}\right)^{\lambda n} \|t^{\alpha} \partial_t^{\alpha} \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{q}} \\ (5.5) \quad &\leq 2^{\frac{\lambda n}{q}} \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x - y| + t}\right)^{\lambda n} \|t^{\alpha} \partial_t^{\alpha} \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{q}} \\ &= 2^{\frac{\lambda n}{q}} g_{\lambda}^{q,*}(f)(x), \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Hence $\|S_{\alpha}^q(f)\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq 2^{\frac{\lambda n}{q}} \|g_{\lambda,\alpha}^{q,*}(f)\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C\|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)}$, for any $f \in L_{\mathbb{B}}^p(\mathbb{R}^n)$. Then, by Theorem 5.3, \mathbb{B} is of Lusin cotype q . \square

Theorem 5.6. *Given a Banach space \mathbb{B} , $1 < q \leq 2$ and $\lambda > 1$, the following statements are equivalent:*

- (i) \mathbb{B} is of Lusin type q .
- (ii) For every (or, equivalently, for some) positive integer n , for every (or, equivalently, for some) $p \in [q, \infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant $C > 0$ such that

$$\|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C \|g_{\lambda,\alpha}^{q,*}(f)\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L_{\mathbb{B}}^p(\mathbb{R}^n).$$

Proof. (i) \Rightarrow (ii). Since \mathbb{B} is of Lusin type q , by Theorem 5.4 and (5.5) we get

$$\|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C \|S_{\alpha}^q(f)\|_{L^p(\mathbb{R}^n)} \leq C \|g_{\lambda,\alpha}^{q,*}(f)\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L_{\mathbb{B}}^p(\mathbb{R}^n).$$

(ii) \Rightarrow (i). By (5.4), we get

$$\|f\|_{L_{\mathbb{B}}^p(\mathbb{R}^n)} \leq C \|g_{\lambda,\alpha}^{q,*}(f)\|_{L^p(\mathbb{R}^n)} \leq C \|g_{\alpha}^q(f)\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L_{\mathbb{B}}^p(\mathbb{R}^n).$$

Then by Theorem B, \mathbb{B} is of Lusin type q . \square

6. PROOF OF THEOREM C

Proof of Theorem C. By Theorem A, Theorem 5.2, Theorem 5.3 and Theorem 5.5, we have (i) \Rightarrow (ii), (i) \Rightarrow (iii) and (i) \Rightarrow (iv).

Let us prove the converse. (ii) \Rightarrow (i). Let $p_0 \in (1, \infty)$. Observe that

$$\begin{aligned} g_{\alpha}^q(f)(x) &= \left(\int_0^{\infty} \|t^{\alpha} \partial_t^{\alpha} \mathcal{P}_t f(x)\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \sup_{j \in \mathbb{Z}^+} \left(\int_{\frac{1}{j}}^j \|t^{\alpha} \partial_t^{\alpha} \mathcal{P}_t f(x)\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{\frac{1}{q}} = \sup_{j \in \mathbb{Z}^+} \|T^j(f)(x, t)\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}, \end{aligned}$$

where $T^j(f)(x, t) = t^{\alpha} \partial_t^{\alpha} \mathcal{P}_t f(x) \chi_{\{\frac{1}{j} < t < j\}}$ is the operator which sends \mathbb{B} -valued functions to $L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})$ -valued functions. It is clear that T^j is bounded from $L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ to $L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^{p_0}(\mathbb{R}^n)$. Let $T_N^j(f)(x) =$

$T^j(f)(x)\chi_{B_N}(x)$, where $B_N = B(0, N)$ is the ball in \mathbb{R}^n , for any $N > 0$. So T_N^j is bounded from $L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ to $L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^{p_0}(B_N)$. Then we have

$$(6.1) \quad \left| \left\{ x \in B_N : \left\| T_N^j(f)(x) \right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})} > \lambda \|f\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\} \right| \leq \frac{1}{\lambda^{p_0} \|f\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)}^{p_0}} \int_{B_N} \left\| T_N^j(f)(x) \right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^{p_0} dx \leq \frac{C}{\lambda^{p_0}}.$$

Let $\mathcal{M} = \left\{ f : f \text{ is } L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})\text{-valued and strong measurable on } B_N \right\}$. In the finite measurable space, (B_N, \mathcal{M}) , we introduce the following topology basis. For any $\varepsilon > 0$, let

$$V_{B_N, \varepsilon} = \left\{ f \in \mathcal{M} : \left| \left\{ x \in B_N : \left\| f(x) \right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})} > \varepsilon \right\} \right| < \varepsilon \right\}.$$

We denote the topology space on B_N by $L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^0(B_N)$. By (6.1), we have

$$\lim_{\lambda \rightarrow \infty} \left| \left\{ x \in B_N : \left\| T_N^j(f)(x) \right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})} > \lambda \|f\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\} \right| = 0.$$

So for any $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that

$$\left| \left\{ x \in B_N : \left\| T_N^j(f)(x) \right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})} > \lambda \|f\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\} \right| < \varepsilon, \quad \lambda \geq \lambda_\varepsilon.$$

Then for ε given above, there exists a constant $\delta_\varepsilon = \frac{\varepsilon}{\lambda_\varepsilon}$, such that for any $\|f\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} < \delta_\varepsilon$ we have

$$\left| \left\{ x \in B_N : \left\| T_N^j(f)(x) \right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})} > \varepsilon \right\} \right| \leq \left| \left\{ x \in B_N : \left\| T_N^j(f)(x) \right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})} > \lambda_\varepsilon \|f\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\} \right| < \varepsilon.$$

This means that $T_N^j(f) \in V_{B_N, \varepsilon}$ for any $f \in L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ with $\|f\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} < \delta_\varepsilon$. Hence T_N^j is continuous from $L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ to $L_{L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t})}^0(B_N)$. Let $U_N = \left\{ T_N^j(f) \right\}_{j=1}^\infty$. Since $g_\alpha^q(f)(x) < \infty$ a.e., U_N is a well defined linear operator from $L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ to $L_{\ell^\infty(L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t}))}^0(B_N)$. As B_N has finite measure, the space $L_{\ell^\infty(L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t}))}^0(B_N)$ is metrizable and complete. Then by the closed graph theorem, the operator U_N is continuous. As $g_{\alpha, N}^q(f)(x) = \left\| T_N^j(f)(x) \right\|_{\ell^\infty(L_{\mathbb{B}}^q(\mathbb{R}_+, \frac{dt}{t}))}$, we get that $g_{\alpha, N}^q$ is continuous from $L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ to $L^0(B_N)$. Therefore for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|\{x \in B_N : |g_\alpha^q(h)(x)| > \varepsilon\}| < \varepsilon, \quad \text{for } \|h\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} < \delta_\varepsilon.$$

In particular, for any $0 < r < \varepsilon$, there exists $\delta_r > 0$ such that

$$|\{x \in B_N : |g_\alpha^q(h)| > r\}| < \varepsilon, \quad \text{for } \|h\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} < \delta_r.$$

Now let g be an element of $L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ with $\|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \neq 0$ and $h = \frac{g}{\|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)}} \frac{\delta_r}{2}$. Then we have

$$\|h\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} < \frac{\delta_r}{2} \text{ and}$$

$$\varepsilon > |\{x \in B_N : |g_\alpha^q(h)| > r\}| > \left| \left\{ x \in B_N : |g_\alpha^q(h)| > \varepsilon \right\} \right| = \left| \left\{ x \in B_N : |g_\alpha^q(g)| > \frac{2\varepsilon \|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)}}{\delta_r} \right\} \right|.$$

Let $\mu_\varepsilon = \frac{2\varepsilon}{\delta_r}$. Then when $\mu \geq \mu_\varepsilon$, we have

$$(6.2) \quad \left| \left\{ x \in B_N : |g_\alpha^q(g)| > \mu \|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\} \right| \leq \left| \left\{ x \in B_N : |g_\alpha^q(g)| > \frac{2\varepsilon \|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)}}{\delta_r} \right\} \right| < \varepsilon.$$

Let $f \in L^1_{\mathbb{B}}(\mathbb{R}^n)$ and $\lambda > 0$, we perform the Calderón–Zygmund decomposition as the sum $f = g + b$ such that $\|g\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)} \leq \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}$ and $\|g\|_{L^\infty_{\mathbb{B}}(\mathbb{R}^n)} \leq 2\lambda$. Then we have

$$(6.3) \quad \|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} \leq (2\lambda)^{\frac{p_0-1}{p_0}} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}^{\frac{1}{p_0}}$$

and

$$(6.4) \quad \left| \left\{ x \in \mathbb{R}^n : |g_\alpha^q(b)(x)| > \frac{\lambda}{2} \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}.$$

Indeed, (6.3) is trivial from the estimates of g . For (6.4), we observe that by Proposition 3.3, $g_\alpha^q(f)$ can be expressed as an $L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$ -norm of a Calderón–Zygmund operator with a regular kernel. In these circumstances, it can be observed that the boundedness of the measure of the set appearing in (6.4) depends only on the kernel of the operator and not on the boundedness of the operator, see [1]. Therefore, by (6.3) and (6.4), we have

$$\begin{aligned} & \left| \left\{ x \in B_N : |g_\alpha^q(f)(x)| > \lambda \right\} \right| \leq \left| \left\{ x \in B_N : |g_\alpha^q(g)(x)| > \frac{\lambda}{2} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n : |g_\alpha^q(b)(x)| > \frac{\lambda}{2} \right\} \right| \\ & = \left| \left\{ x \in B_N : |g_\alpha^q(g)(x)| > \frac{\lambda}{2} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n : |g_\alpha^q(b)(x)| > \frac{\lambda}{2} \right\} \right| \\ & \leq \left| \left\{ x \in B_N : |g_\alpha^q(g)(x)| > \frac{\lambda^{\frac{1}{p_0}}}{2^{2-\frac{1}{p_0}} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}^{\frac{1}{p_0}}} \|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} \right\} \right| + \frac{C}{\lambda} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)} \\ & = \left| \left\{ x \in B_N : |g_{\alpha,N}^q(g)(x)| > \frac{\lambda^{\frac{1}{p_0}}}{2^{2-\frac{1}{p_0}} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}^{\frac{1}{p_0}}} \|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} \right\} \right| + \frac{C}{\lambda} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}. \end{aligned}$$

Now, given $\varepsilon > 0$ we perform the Calderón–Zygmund decomposition with λ such that $\lambda^{\frac{1}{p_0}} > 2^{2-\frac{1}{p_0}} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}^{\frac{1}{p_0}} \mu_\varepsilon$. Then, by (6.2), we have

$$\begin{aligned} & \left| \left\{ x \in B_N : |g_\alpha^q(f)(x)| > \lambda \right\} \right| \leq \left| \left\{ x \in B_N : |g_\alpha^q(g)(x)| > \mu_\varepsilon \|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} \right\} \right| + \frac{C}{\lambda} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)} \\ & \leq \varepsilon + \frac{C}{\lambda} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}. \end{aligned}$$

This clearly implies $g_\alpha^q(f)(x) < \infty$ a.e. $x \in \mathbb{R}^n$, for any $f \in L^1_{\mathbb{B}}(\mathbb{R}^n)$. We apply Theorem 1.3 and get the result.

To prove that (iii) \Rightarrow (i), we can use the same argument as above but with a very small modification. We only need note that

$$\begin{aligned} S_\alpha^q(f)(x) &= \left(\iint_{\Gamma(x)} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{q}} = \left(\int_0^\infty \int_{|y-x|<t} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^n} \right)^{\frac{1}{q}} \\ &= \sup_{j \in \mathbb{Z}^+} \left(\int_{\frac{1}{j}}^j \int_{|y-x|<t} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy dt}{t^n} \right)^{\frac{1}{q}} = \sup_{j \in \mathbb{Z}^+} \|T^j(f)(x, t)\|_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})}, \end{aligned}$$

where $T^j(f)(x, t) = \int_{|y-x|<t} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy}{t^n} \chi_{\{\frac{1}{j} < t < j\}}$ is the operator which sends \mathbb{B} -valued functions to $L^q(\mathbb{R}_+, \frac{dt}{t})$ -valued functions. And T^j is bounded from $L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ to $L^{p_0}_{L^q(\mathbb{R}_+, \frac{dt}{t})}(\mathbb{R}^n)$, $1 < p_0 < \infty$ also. Now we can continue the proof as in the case of g_α^q .

(iv) \Rightarrow (i). Assuming that $g_{\lambda,\alpha}^{q,*}(f)(x) < \infty$ a.e. $x \in \mathbb{R}^n$, by (5.5) we know that $S_\alpha^q(f)(x) \leq C g_{\lambda,\alpha}^{q,*}(f)(x) < \infty$ a.e. $x \in \mathbb{R}^n$. Then by (iii) \Rightarrow (i), \mathbb{B} is of Lusin cotype q . \square

7. UMD SPACES

Now we give the proof of Theorem D. Clearly it is enough to prove (ii) \Rightarrow (i). Let $1 < p_0 < \infty$ and assume that $\lim_{\varepsilon \rightarrow 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$ exists a.e. $x \in \mathbb{R}$ for any $f \in L_{\mathbb{B}}^{p_0}(\mathbb{R})$. Then the maximal operator $H^*f(x) = \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \right\|_{\mathbb{B}}$ is finite a.e. $x \in \mathbb{R}$. Our idea is to apply the method developed in the proof of (ii) \Rightarrow (i) of Theorem C. However, we cannot apply it directly since H^* can't be expressed as a norm of a Calderón-Zygmund operator with a regular kernel. Let φ be a smooth function such that $\chi_{[\frac{3}{2}, \infty)} \leq \varphi \leq \chi_{[\frac{1}{2}, \infty)}$. Consider the operator $H_{\varphi}^*f(x) = \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}} \varphi\left(\frac{|x-y|}{\varepsilon}\right) f(y) dy \right\|_{\mathbb{B}}$. It can be easily checked that

$$(7.1) \quad |H_{\varphi}^*f(x) - H^*f(x)| \leq CM(\|f\|_{\mathbb{B}})(x), \quad \text{a.e. } x \in \mathbb{R},$$

where M denotes the Hardy-Littlewood maximal function. Therefore, the operator $H_{\varphi}^*f(x) < \infty$, a.e. $x \in \mathbb{R}$. Observe that this operator can be expressed as

$$H_{\varphi}^*f(x) = \left\| \left\{ \int_{\mathbb{R}} \varphi\left(\frac{|x-y|}{\varepsilon}\right) f(y) dy \right\}_{\varepsilon} \right\|_{\ell_{\mathbb{B}}^{\infty}}.$$

It is well known that the last operator can be viewed as the $\ell_{\mathbb{B}}^{\infty}$ -norm of a Calderón-Zygmund operator with regular kernel. Now we are in the situation of the proof of part (ii) \Rightarrow (i) of Theorem C and with some obvious changes we get

$$\lim_{\lambda \rightarrow \infty} |\{x \in B_N : |H_{\varphi}^*(f)(x)| > \lambda\}| = 0, \quad \forall f \in L_{\mathbb{B}}^1(\mathbb{R}), \quad N > 0.$$

In particular, this implies that the operator H_{φ}^* maps $L_{\mathbb{B}}^1(\mathbb{R})$ into $L^0(\mathbb{R})$. By (7.1) and the fact that M maps $L_{\mathbb{B}}^1(\mathbb{R})$ into weak- $L^1(\mathbb{R})$ for every Banach space \mathbb{B} , H^* maps $L_{\mathbb{B}}^1(\mathbb{R})$ into $L^0(\mathbb{R})$. Now we can apply the following lemma.

Lemma 7.1. [7, Lemma 7.3] *Let \mathbb{B} be a Banach space. Then every translation and dilation invariant continuous sublinear operator $T : L_{\mathbb{B}}^1(\mathbb{R}^n) \rightarrow L^0(\mathbb{R}^n)$ is of weak type $(1, 1)$.*

Then we get $H^* : L_{\mathbb{B}}^1(\mathbb{R}) \rightarrow \text{weak-}L^1(\mathbb{R})$ which implies that the Banach space \mathbb{B} is UMD. This ends the proof of Theorem D.

Remark 7.2. The above thoughts can be apply to the following general situation.

Given two Banach spaces $\mathbb{B}_1, \mathbb{B}_2$ and $1 \leq p < \infty$, let $K(x, y) \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ be a regular Calderón-Zygmund kernel. Define $T_{\varepsilon}f(x) = \int_{|x-y|>\varepsilon} K(x, y)f(y)dy$ and

$$Sf(x) = \lim_{\varepsilon \rightarrow 0^+} T_{\varepsilon}f(x), \quad x \in \mathbb{R}^n.$$

Then the following statements are equivalent:

- For any $p \in (1, \infty)$, the operator S maps $L_{\mathbb{B}_1}^p(\mathbb{R}^n)$ into $L_{\mathbb{B}_2}^p(\mathbb{R}^n)$.
- For any (or, equivalently, for some) $p \in (1, \infty)$, the maximal operator $S^*f(x) = \sup_{\varepsilon>0} \|T_{\varepsilon}f(x)\|_{\mathbb{B}_2} < \infty$, a.e. $x \in \mathbb{R}^n$ for every $f \in L_{\mathbb{B}_1}^p(\mathbb{R}^n)$.

In other words, the following statement

“There exists a number $p_0 \in [1, \infty)$ such that $\|Tf(x)\|_{\mathbb{B}_2} < \infty$ a.e. $x \in \mathbb{R}^n$, for every $f \in L_{\mathbb{B}_1}^{p_0}(\mathbb{R}^n)$.” could be added to the list of those statements in Remark 3.2, after an appropriated description of T .

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